Mathematics

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http://math.yale.edu
M.S., M.Phil., Ph.D.

Chair
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Professors Richard Beals (Emeritus), Jeffrey Brock, Andrew Casson (Emeritus), Ronald Coifman, Igor Frenkel, Howard Garland (Emeritus), Alexander Goncharov, Roger Howe (Emeritus), Peter Jones, Gil Kalai (Adjunct), Ivan Losev, Alexander Lubotzky (Adjunct), Gregory Margulis, Yair Minsky, Vincent Moncrief (Physics), Hee Oh, Sam Payne, Nicholas Read (Physics; Applied Physics), Vladimir Rokhlin (Computer Science), Wilhelm Schlag, George Seligman (Emeritus), Daniel Spielman (Computer Science), Van Vu, John Wettlaufer (Geology & Geophysics; Physics), Gregg Zuckerman

Associate Professor Yifeng Liu

Assistant Professor Stefan Steinerberger

Fields of Study
Fields include real analysis, complex analysis, functional analysis, classical and modern harmonic analysis; linear and nonlinear partial differential equations; dynamical systems and ergodic theory; geometric analysis; kleinian groups, low dimensional topology and geometry; differential geometry; finite and infinite groups; geometric group theory; finite and infinite dimensional Lie algebras, Lie groups, and discrete subgroups; representation theory; automorphic forms, L-functions; algebraic number theory and algebraic geometry; mathematical physics, relativity; numerical analysis; combinatorics and discrete mathematics.

Special Requirements for the Ph.D. Degree
All students are required to: (1) complete eight term courses at the graduate level, at least two with Honors grades; (2) pass qualifying examinations on their general mathematical knowledge; (3) submit a dissertation prospectus; (4) participate in the instruction of undergraduates; (5) be in residence for at least three years; and (6) complete a dissertation that clearly advances understanding of the subject it considers. The normal time for completion of the Ph.D. program is five years. Requirement (1) should be completed by the end of the second year. A sequence of three qualifying examinations (algebra and number theory, real and complex analysis, topology) is offered each term, at intervals of about one month. All qualifying examinations must be taken by the end of the third term. The thesis is expected to be independent work, done under the guidance of an adviser. This adviser should be contacted not long after the student passes the qualifying examinations. A student is admitted to candidacy after completing requirements (1)–(5) and obtaining an adviser.

In addition to all other requirements, students must successfully complete MATH 991, Ethical Conduct of Research, prior to the end of their first year of study. This requirement must be met prior to registering for a second year of study.

Honors Requirement
Students must meet the Graduate School’s Honors requirement by the end of the fourth term of full-time study.

Teaching
Teaching experience is integral to graduate education at Yale. Therefore, most Mathematics students are required to assist in teaching during five terms. Students in years one and two serve as tutors and graders in undergraduate mathematics courses during one term per year. The department also offers a required teaching practicum in year two. In years three through five, students normally teach one section of calculus or its equivalent during one term per year. Students receiving external fellowships may petition for a waiver of teaching while receiving external funding in place of University funding, but they are still required to teach one section of calculus or its equivalent for a minimum of two terms over the course of their program.

Master’s Degrees
M.Phil. In addition to the Graduate School’s Degree Requirements (see under Policies and Regulations), a student must undertake a reading program of at least two terms’ duration in a specific significant area of mathematics under the supervision of a faculty adviser and demonstrate command of the material studied during the reading period at a level sufficient for teaching and research.

M.S. (en route to the Ph.D.) A student must complete six term courses with at least one Honors grade, perform adequately on the general qualifying examination, and be in residence at least one year. The M.S. degree is conferred only en route to the Ph.D.; there is no separate master’s program in Mathematics.
COURSES

**MATH 500a, Modern Algebra I**  Yifeng Liu
A survey of algebraic constructions and theories at a sophisticated level. Topics include categorical language, free groups and other free objects in categories, general theory of rings and modules, artinian rings, and introduction to homological algebra.

**MATH 520a, Measure Theory and Integration**  Arie Levit
Construction and limit theorems for measures and integrals on general spaces; product measures; Lp spaces; integral representation of linear functionals.

**MATH 544a, Introduction to Algebraic Topology I**  Yair Minsky
A one-term graduate introductory course in algebraic topology. We discuss algebraic and combinatorial tools used by topologists to encode information about topological spaces. Broadly speaking, we study the fundamental group of a space, its homology, and its cohomology. While focusing on the basic properties of these invariants, methods of computation, and many examples, we also see applications toward proving classical results. These include the Brouwer fixed-point theorem, the Jordan curve theorem, Poincaré duality, and others. The main text is Allen Hatcher's *Algebraic Topology*, which is available for free on his website.

**MATH 573b, Algebraic Number Theory**  Asher Auel
Structure of fields of algebraic numbers (solutions of polynomial equations with integer coefficients) and their rings of integers; prime decomposition of ideals and finiteness of the ideal class group; completions and ramification; adeles and ideles; zeta functions.

**MATH 600a, Topics in Homogeneous Dynamics and Geometry**  Hee Oh
We discuss a variety of topics in homogeneous dynamics, especially in relation to hyperbolic manifolds of infinite volume. Possible topics include: (1) basics in hyperbolic geometry; (2) constructions of (non-) arithmetic lattices and rigid acylindrical hyperbolic manifolds; (3) mixing of the frame flow, and applications to counting and equidistribution problems; (4) the spectrum of the Laplacian, expanders, and effective mixing; (5) orbit closures on the frame bundle of a hyperbolic 3-manifold; (6) Margulis function and its applications; (7) linearization and limit distribution of orbits of unipotent flows; (8) orbit closures on the frame bundle of a hyperbolic n-manifold.

**MATH 608a, Introduction to Arithmetic Geometry**  Asher Auel
This course explores some of the major themes in arithmetic geometry. Topics include Galois cohomology and descent, principal homogeneous spaces, quadratic forms and the Brauer group, Milnor K-theory, as well as the existence of rational points. Of particular interest are varieties over finite fields, number fields, and function fields. Prerequisites: experience with algebra and Galois theory is necessary. Some exposure to algebraic geometry is useful, but the course may be taken in conjunction with a beginning algebraic geometry course.

**MATH 660a, Classical Statistical Thermodynamics**  John Wettlaufer
Classical thermodynamics is derived from statistical thermodynamics. Using the multi-particle nature of physical systems, we derive ergodicity, the central limit theorem, and the elemental description of the second law of thermodynamics. We then develop kinetics, transport theory, and reciprocity from the linear thermodynamics of irreversible processes. Topics of focus include Onsager reciprocal relations, the Fokker-Planck equation, stability in the sense of Lyapunov, and time invariance symmetry. We explore phenomena that are of direct relevance to astrophysical and geophysical settings. No quantum mechanics is necessary as a prerequisite.

**MATH 670a, Topics on Random Graphs**  Mathias Schacht
We discuss a variety of topics on the theory of random graphs. We introduce the standard models of random graphs and focus on the threshold phenomenon for graph properties. For many interesting and natural graph properties, the probability for a random graph to enjoy the property moves from close to 0 to close to 1 in a relatively small interval in terms of the given density of the random graph; we investigate this for the properties of (1) containing fixed size or spanning subgraphs (like a perfect matching or a Hamiltonian cycle); (2) chromatic number; (3) transfer of classical theorems in extremal combinatorics. This course is open to students from Statistics and Computer Science as well. Yale College juniors and seniors are also welcome. Some background in discrete probability and graph theory is helpful, but the course is self-contained.

**MATH 674b, Extremal Combinatorics**  Patrick Devlin
The course is a standalone introduction to extremal combinatorics. We focus on algebraic and combinatorial techniques (e.g., combinatorial Nullstellensatz, tensor constructions, duality, shifting) as applied to a variety of discrete settings including hypergraphs and set systems, designs, graphs, posets, and arithmetic combinatorics. Time permitting, we may also discuss topics from eigenvalue methods, entropy, linear programming, finite geometry, or coding theory. No prerequisites. Interested undergraduates are encouraged to contact the instructor.

**MATH 675a, Numerical Methods for Partial Differential Equations**  Vladimir Rokhlin
(7) Numerical methods for hyperbolic PDEs. Finite differences, finite elements, Richardson and deferred corrections, time-invariant problems and Fourier transform.

MATH 676a, Hyperbolic Geometry and Related Topics  
Kenneth Bromberg

This course covers some topics in quasi-conformal maps, Teichmüller theory, and hyperbolic 3-manifolds. In the first part of the course we aim to be reasonably detailed, and the material should be accessible to anyone who has completed standard first-year graduate courses. We begin by developing some basics about quasi-conformal maps: geometric and analytic definitions, compactness theorems, extremal length, the measurable Riemann mapping theorem (Glusky’s proof via Fourier series), Schwarzian derivatives. The end goal of the first part of the course is to explain how the Bers embedding gives a canonical complex structure on Teichmüller space. In the second part of the course we discuss hyperbolic 3-manifolds. In particular, the Bers embedding identifies Teichmüller space with a family of quasi-Fuchsian hyperbolic 3-manifolds. Hyperbolic 3-manifolds can be studied from many different points of view; our approach is via differential geometry. Our main goal is to define and study Krasnov-Schlenker’s notion of renormalized volume. This defines a smooth function on the Bers embedding, and we see how studying this function reveals geometric information about the hyperbolic manifolds. The material is more advanced, and we black box a certain amount of background material on hyperbolic 3-manifolds.

MATH 701b, Topics in Analysis  
Peter Jones

MATH 724b, Heat Kernel and Analysis on Manifolds  
Stefan Steinerberger

Topics include Laplace operator on Riemannian manifolds, heat equation, maximum principles and regularity theory, spectral properties, the distance function, Gaussian estimates, Davies-Gaffney estimates, Green function, Ultracontractive estimates, pointwise Gaussian estimates. The goal is to go through Heat Kernel and Analysis on Manifolds by Alexander Grigor’yan.

MATH 754b, Infinite-Dimensional Lie Algebras and Applications  
Oleksandr Tsymbaliuk

The basic course studying infinite-dimensional Lie algebras, the key two examples of which are Virasoro Lie algebra and affine Kac-Moody algebras. Their theory as well as various applications to other areas are discussed.

MATH 755b, Vertex Algebras and Representations  
Fei Qi

This course studies the vertex algebras and their representation theory. We follow two textbooks by, respectively, Frenkel-Lepowsky-Meurman and Lepowsky-Li. As we mainly use infinite dimensional Lie algebras to construct examples of vertex algebras, it is highly recommended that students also take a course on infinite Lie algebras. Depending on the interests of students, we can also include advanced topics such as intertwining operators, modular tensor category, cohomology theory, and application in conformal field theory, etc.

MATH 756b, Geodesic Currents and Counting Problems  
Caglar Uyanik

Geodesic currents are measure theoretic generalizations of closed curves on hyperbolic surfaces, and they play an important role, among other things, in the study of Teichmüller spaces and mapping class groups. The goal of this course is to study groundbreaking work of Mirzakhani on counting closed geodesics on hyperbolic surfaces [Mir10, Mir16] and its generalizations [EPS16, ES16, RS17] using geodesic currents. We first review background material such as hyperbolic geometry, mapping class groups, Teichmüller and moduli spaces, measured laminations, and train tracks. We then move to geodesic currents and discuss applications to counting problems on hyperbolic surfaces. The main sources for the first part are CB88, FM11, and PH16, and the second part mainly draws from Bon86, Bon88, Miro8, Mir16, EPS16, ES16, EU18, and RS17.

Prerequisite: the contents of Algebraic Topology 1.

MATH 792b, Stability and Invariant Random Subgroups  
Arie Levit and Alexander Lubotzky

The course focuses on stability, which is a novel notion in group theory. A group G is stable if any almost homomorphism from G to another target group is close to an actual homomorphism, in a controlled way. There has been recent interest in studying stability, partially motivated by the relation to sofic groups. Namely, a stable, nonresidually finite group cannot be sofic: that is, it cannot be approximated in a certain very flexible way by finite groups. We discuss in detail a paper by Becker, Lubotzky, and Thom that gives a necessary and sufficient condition for the stability of amenable groups. The condition is formulated in terms of invariant random subgroups, another novel notion in group theory that is studied in this course. The seemingly unexpected relation between stability and invariant random subgroups turns out to be quite natural. If time permits, we discuss stability in several classes of amenable and non-amenable groups, e.g., coming from geometric group theory. Prerequisite: an undergraduate course in group theory.

MATH 801a, Cocycles, Lyapunov Exponents, and Spectral Theory  
Wilhelm Schlag

We develop cocycles over an ergodic base, prove the basic ergodic theorems (Kingman), and discuss Lyapunov exponents, Osseldt’s theorem, and Furstenberg’s theorem in the random case, as well as its higher-dimensional version. We connect these results to the spectral theory of Schrödinger operators with potentials defined by an ergodic process, introduce the avalanche principle, and establish Hölder regularity of the Lyapunov exponent in the energy.

MATH 802a, Mathematics of Two-Dimensional Quantum Gravity and Topological Strings  
PhilSang Yoo

In this course, we discuss the interconnection between several important subjects in mathematics, including moduli theory, integrable hierarchies, graphs on surfaces, complex geometry, and singularity theory, using certain ideas from theoretical physics as a main thread. The first part of the course is mainly devoted to studying moduli spaces of Riemann surfaces, which are fundamental objects in complex analysis, geometric topology, algebraic geometry, arithmetic geometry, and mathematical physics. We introduce their Deligne-Mumford compactifications, consider some of their natural cohomology classes, and study the intersection numbers between them. This part culminates with discussion of Witten’s conjecture on a two-dimensional quantum gravity model that a certain generating function of the intersection numbers should satisfy the differential equations of the KdV hierarchy. For this, we provide a quick overview
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of integrable hierarchies and then sketch Kontsevich’s proof of the conjecture. In the second part of the course, we discuss mathematics of
topological string theory, especially of what is usually referred to as the B-model in the subject. Mathematically speaking, this amounts to
the study of complex algebraic geometry and Hodge theory of Calabi-Yau manifolds on the one hand and singularity theory on the other.
Our aim is to understand both subjects in a uniform and coherent way, as motivated by ideas of topological string theory. Our treatment
is built on the rigorous mathematical frameworks developed by several people in and around the subject of mirror symmetry, including
Barannikov, Dubrovin, Givental, Kontsevich, and K. Saito. Class material is entirely mathematical and in particular does not assume any
background on physics; physics will be used only as a guiding principle for the choice of topics and a presentation. On the other hand,
the course assumes basic knowledge of complex analysis, algebraic topology, differential geometry, and algebraic geometry; references are
provided for those who are less prepared but willing to spend extra time to follow the course.

MATH 827b, Lang Teaching Seminar  Marketa Havlickova
This course prepares graduate students for teaching calculus classes. It is a mix of theory and practice, with topics such as preparing
classes, presenting new concepts, choosing examples, encouraging student participation, grading fairly and effectively, implementing
active learning strategies, and giving and receiving feedback. Open only to mathematics graduate students in their second year.

MATH 845a, Introduction to Algebraic Geometry  Kalina Mincheva

MATH 991a / CPSC 991a, Ethical Conduct of Research  Alexander Goncharov